

# ON EXISTENCE OF INFINITE PRIMES AND INFINITE TWIN PRIMES

Maurice Margenstern\* and Yaroslav D. Sergeyev<sup>† ‡</sup>

## Abstract

The twin primes conjecture is a very old problem. Tacitly it is supposed that the primes it deals with are finite. In the present paper we consider three problems that are not related to finite primes but deal with infinite integers. The main tool of our investigation is a numeral system proposed recently that allows one to express various infinities and infinitesimals easily and by a finite number of symbols. The problems under consideration are the following and for all of them we give affirmative answers: (i) do infinite primes exist? (ii) do infinite twin primes exist? (ii) is the set of infinite twin primes infinite? Examples of these three kinds of objects are given.

**Key Words:** Infinite primes, infinite twin primes, infinite sets.

## 1 Introduction

The research on twin primes number has given rise to an important number of works (see, for example, [6] and references given therein). In this paper, we study the problem of the existence of infinite primes and infinite twin primes. So, our paper brings in no new light on the traditional twin primes conjecture. Let us mention a few works which can be considered as precursors in some sense and might be thought of an exotic character by pure mathematicians. As an example of such an exotic issue, we can quote the existence of other natural families of numbers whose distribution is alike that of primes. There is an example of such a family for which an analogue of the twin primes can be formulated and was indeed proved, see [14]. Now, what we can consider as precursors for us are more connected with logical problems. The first paper in this direction is [11], where

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\*Maurice Margenstern, Ph.D., is Professor Emeritus at the Université de Lorraine, e-mail: [margenstern@gmail.com](mailto:margenstern@gmail.com)

<sup>†</sup>Yaroslav D. Sergeyev, Ph.D., D.Sc., D.H.C., is Distinguished Professor at the University of Calabria, Rende, Italy. He is also Professor (a part-time contract) at the Lobachevsky State University, Nizhni Novgorod, Russia and Affiliated Researcher at the Institute of High Performance Computing and Networking of the National Research Council of Italy, e-mail: [yaro@si.dimes.unical.it](mailto:yaro@si.dimes.unical.it)

<sup>‡</sup>Corresponding author.

the possibility of infinite prime numbers of the form we consider in this paper was investigated without reaching definite results. The second paper is [19] where the author constructs a model of a theory in which it is possible to prove the twin prime conjecture. There is an important difference in this paper with our result in this sense that the theory considered by this author assumes a weak induction axiom while here, we have no induction at all on infinite integers. The form of the twin primes in that paper has some similarity with that of our infinite twin primes, although the paper appeals to higher results in algebra and analysis which is not at all the case of our work.

A numeral system introduced recently in [20, 21] for performing computations with infinities and infinitesimals is used here to study the problem of the existence of infinite primes and infinite twin primes. It should be mentioned that this computational methodology is not related to non-standard analysis of Robinson and has a strong applied character. In fact, the Infinity Computer working numerically with a variety of infinite and infinitesimal numbers has been introduced (see the patent [23]).

In order to see the place of the new approach in the historical panorama of ideas dealing with infinite and infinitesimal, see [10, 12, 13, 15, 18, 22, 24, 32]. In particular, connections of the new approach with bijections is studied in [15] and metamathematical investigations on the theory can be found in [13]. Among the applications where the new approach has been successfully used we can mention the following: percolation and biological processes (see [7, 8, 34, 28]), hyperbolic geometry (see [16, 17]), numerical differentiation and optimization (see [1, 26, 36]), infinite series (see [9, 22, 27, 35]), the first Hilbert problem, Turing machines, and lexicographic ordering (see [24, 31, 32, 33]), cellular automata (see [2, 3, 4]), ordinary differential equations (see [30]), etc.

The rest of the paper has the following structure. The next Section contains a brief informal description of the numeral system allowing one to express different infinities and infinitesimals in a unique framework (see [21, 25, 29] for a detailed discussion). Section 3 presents main results of the paper.

## 2 Infinities and infinitesimals expressed in grossone-based numerals

Let us consider a study published in *Science* (see [5]) where there is a description of a primitive tribe living in Amazonia - Pirahã - that uses a very simple numeral system<sup>1</sup> for counting: one, two, many.

For Pirahã, all quantities larger than two are just ‘many’ and such operations as

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<sup>1</sup> We remind that *numeral* is a symbol or a group of symbols that represents a *number*. The difference between numerals and numbers is the same as the difference between words and the things they refer to. A *number* is a concept that a *numeral* expresses. The same number can be represented by different numerals. For example, the symbols ‘3’, ‘three’, and ‘III’ are different numerals, but they all represent the same number.

2+2 and 2+1 give the same result, i.e., ‘many’. Using their weak numeral system Pirahã are not able to see numbers 3, 4, etc., to execute arithmetical operations with them, and, in general, to say anything about these numbers because in their language there are neither words nor concepts for that. Moreover, the weakness of their numeral system leads to such results as

$$\text{‘many’} + 1 = \text{‘many’}, \quad \text{‘many’} + 2 = \text{‘many’},$$

which are very familiar to us in the context of views on infinity used in the traditional calculus

$$\infty + 1 = \infty, \quad \infty + 2 = \infty.$$

This analogy advises that our difficulty in working with infinity is not connected to the nature of infinity but is just a result of inadequate numeral systems used to express numbers. In fact, numeral systems strongly influence our capabilities to describe physical and mathematical objects. For instance, Roman numeral system has no numeral to express 0. As a consequence, the expression III-X in this numeral system is an indeterminate form. Moreover, any assertion regarding negative numbers and zero cannot be formulated using Roman numerals because there are no symbols corresponding to these concepts in this concrete numeral system.

The numeral system proposed in [21, 25, 29] is based on an infinite unit of measure expressed by the numeral ① called *grossone* and introduced as the number of elements of the set  $\mathbb{N}$  of natural numbers (a clear difference with non-standard analysis can be seen immediately since non-standard infinite numbers are not connected to concrete infinite sets and do not belong to  $\mathbb{N}$ ). Other symbols dealing with infinities and infinitesimals ( $\infty$ , Cantor’s  $\omega$ ,  $\aleph_0$ ,  $\aleph_1$ , ..., etc.) are not used together with ①. Similarly, when the positional numeral system and the numeral 0 expressing zero had been introduced, symbols V, X, and other symbols from the Roman numeral system had not been involved.

Notice that people very often do not pay a great attention to the distinction between numbers and numerals (in this occasion it is necessary to recall constructivists who studied this issue), many theories dealing with infinite and infinitesimal quantities have a symbolic (not numerical) character. For instance, many versions of non-standard analysis are symbolic, since they have no numeral systems to express their numbers by a finite number of symbols (the finiteness of the number of symbols is necessary for organizing numerical computations). Namely, if we consider a finite  $n$  then it can be taken  $n = 7$ , or  $n = 108$  or any other numeral used to express finite quantities and consisting of a finite number of symbols. In contrast, if we consider a non-standard infinite  $m$  then it is not clear which numerals can be used to assign a concrete value to  $m$ . One of the important differences between the new approach and non-standard analysis consists of the fact that the new numeral system allows us to assign concrete values to infinities (and infinitesimals) as it happens with finite values. In fact, we can assign  $m = \textcircled{1}$ ,  $m = 3\textcircled{1} - 2$  or to use any other infinite numeral involving grossone to give a numerical value to  $m$  (see [21, 25, 29] for a detailed discussion).

The numeral  $\textcircled{1}$  allows one to construct different numerals expressing different infinities and infinitesimals and to execute numerical computations with all of them. As a result, in occasions requiring infinities and infinitesimals indeterminate forms and various kind of divergence are not present when one works with any (finite, infinite, or infinitesimal) numbers expressible in the new numeral system and it becomes possible to execute arithmetical operations with a variety of different infinities and infinitesimals. For example, for  $\textcircled{1}$  and  $\textcircled{1}^{3.1}$  (that are examples of infinities) and  $\textcircled{1}^{-1}$  and  $\textcircled{1}^{-3.1}$  (that are examples of infinitesimals) it follows

$$\begin{aligned}
0 \cdot \textcircled{1} &= \textcircled{1} \cdot 0 = 0, \quad \textcircled{1} - \textcircled{1} = 0, \quad \frac{\textcircled{1}}{\textcircled{1}} = 1, \quad \textcircled{1}^0 = 1, \quad 1^{\textcircled{1}} = 1, \quad 0^{\textcircled{1}} = 0, \quad (1) \\
0 \cdot \textcircled{1}^{-1} &= \textcircled{1}^{-1} \cdot 0 = 0, \quad \textcircled{1}^{3.1} > \textcircled{1}^1 > 1 > \textcircled{1}^{-1} > \textcircled{1}^{-3.1} > 0, \\
\textcircled{1}^{-1} - \textcircled{1}^{-1} &= 0, \quad \frac{\textcircled{1}^{-1}}{\textcircled{1}^{-1}} = 1, \quad \frac{5 + \textcircled{1}^{-3.1}}{\textcircled{1}^{-3.1}} = 5\textcircled{1}^{3.1} + 1, \quad (\textcircled{1}^{-1})^0 = 1, \\
\textcircled{1} \cdot \textcircled{1}^{-1} &= 1, \quad \textcircled{1} \cdot \textcircled{1}^{-3.1} = \textcircled{1}^{-2.1}, \quad \frac{\textcircled{1}^{3.1} + 4\textcircled{1}}{\textcircled{1}} = \textcircled{1}^{2.1} + 4, \\
\frac{\textcircled{1}^{3.1}}{\textcircled{1}^{-3.1}} &= \textcircled{1}^{6.2}, \quad (\textcircled{1}^{3.1})^0 = 1, \quad \textcircled{1}^{3.1} \cdot \textcircled{1}^{-1} = \textcircled{1}^{2.1}, \quad \textcircled{1}^{3.1} \cdot \textcircled{1}^{-3.1} = 1.
\end{aligned}$$

It follows from (1) that  $\textcircled{1}^0 = 1$ , therefore, a finite number  $a$  can be represented in the new numeral system simply as  $a\textcircled{1}^0 = a$ , where the numeral  $a$  itself can be written down by any convenient numeral system used to express finite numbers. The simplest infinitesimal numbers are represented by numerals having only negative finite powers of  $\textcircled{1}$  (e.g.,  $50.1\textcircled{1}^{-10.2} + 16.38\textcircled{1}^{-20.3}$ , see also examples above). Notice that all infinitesimals are not equal to zero. In particular,  $\frac{1}{\textcircled{1}} > 0$  because it is a result of division of two positive numbers. We shall not speak more about infinitesimals here since they are not used in the present paper.

It should be mentioned that in certain cases  $\textcircled{1}$ -based numerals allow us to execute a finer analysis of infinite objects than traditional tools allow us to do. For instance, it becomes possible to measure certain infinite sets and to see, e.g., that the sets of even and odd numbers have  $\textcircled{1}/2$  elements each. The set  $\mathbb{Z}$  of integers has  $2\textcircled{1} + 1$  elements ( $\textcircled{1}$  positive elements,  $\textcircled{1}$  negative elements, and zero). Within the countable sets and sets having cardinality of the continuum (see [12, 24, 25]) it becomes possible to distinguish infinite sets having different number of elements expressible in the numeral system using grossone and to see that, for instance,

$$\begin{aligned}
\frac{\textcircled{1}}{2} &< \textcircled{1} - 1 < \textcircled{1} < \textcircled{1} + 1 < 2\textcircled{1} + 1 < 2\textcircled{1}^2 - 1 < 2\textcircled{1}^2 < 2\textcircled{1}^2 + 1 < \\
2\textcircled{1}^2 + 2 &< 2^{\textcircled{1}} - 1 < 2^{\textcircled{1}} < 2^{\textcircled{1}} + 1 < 10^{\textcircled{1}} < \textcircled{1}^{\textcircled{1}} - 1 < \textcircled{1}^{\textcircled{1}} < \textcircled{1}^{\textcircled{1}} + 1.
\end{aligned}$$

It is important to stress that the new approach does not contradict Cantor's results. The situation is similar to what happens when one uses a microscope with two different lenses: the first of them is weak and allows one to see the object of the

observation as two dots and another lens is stronger and allows the observer to see instead of the first dot 10 dots and instead of the second dot 32 dots. Both lenses give two correct answers having different accuracies. Analogously, both approaches, Cantor's and the new one, give correct answers but the accuracy of the answers is different. Cantor's tools say that the sets of even, odd, natural, and integer numbers have the same cardinality  $\aleph_0$ . This answer is correct with the precision that cardinal numbers have. However, the fact that they all have the same cardinality can be viewed also as the accuracy of the used instrument is too low to see that these sets have different numbers of elements. The new numeral system allows us to see these differences among sets having cardinality  $\aleph_0$  and among sets having cardinality of the continuum, as well (see [12, 15, 24, 25, 29] for a detailed discussion including also the one-to-one correspondence issues).

We conclude this brief informal introduction by mentioning that properties of grossone are described by the Infinite Unit Axiom (see [24, 25, 29]) that is added to axioms for real numbers. In the context of the present paper two issues postulated by the Axiom are important for us: grossone is an infinite number; (ii) grossone is divisible by any finite integer. Notice that grossone is not the only number enjoying the latter property. In fact, zero is also divisible by any finite integer.

### 3 Infinite primes and twin primes

We need some definitions and conventions to continue our study. First, in the further consideration we use the following representation of an infinite number  $c$  where its infinite part is separated from its finite part:  $c = c_1 + c_2$ . In this separation,  $c_1$  is infinite and is expressed by numerals involving  $\textcircled{1}$  and  $c_2$  is finite and is represented by numerals used to write down finite numbers. Note that such a representation is not unique. For instance, the number  $k = 1.7\textcircled{1} - 1.5$  can be decomposed as  $k_1 = 1.7\textcircled{1}$ ,  $k_2 = -1.5$ , or as  $\tilde{k}_1 = 1.7\textcircled{1} - 1$ ,  $\tilde{k}_2 = -0.5$ , or in some other way. Clearly, this decomposition becomes unique if we require that the part  $c_1$  does not contain any part expressed by finite numerals only. In our example with the decomposition of the number  $k$  we have its unique decomposition  $k_1 = 1.7\textcircled{1}$ ,  $k_2 = -1.5$ .

Then, infinite numbers that do not contain finite parts are called *purely infinite*. In other words, this means that in their unique decomposition they have  $c_2 = 0$ . Infinite numbers having their infinite part  $c_1$  including more than one infinite part represented by different powers of  $\textcircled{1}$  are called *compound*. Numbers that are not compound are called *simple*. For instance, the numbers  $\textcircled{1} - 3\textcircled{1}^{\frac{1}{2}}$  and  $\textcircled{1}^2 + \textcircled{1} + 3.5$  are both compound; the former is purely infinite while the latter is not. The numbers  $\frac{\textcircled{1}}{2}$  and  $\textcircled{1}^2 + 1$  are examples of simple infinite numbers; again the former is purely infinite while the latter is not. Finally, if a finite or infinite number  $c$  is the square of an integer  $d$ , i.e.,  $c = d^2$ , we say hereinafter simply that  $c$  is a square.

**Lemma 3.1.** *There exist purely infinite simple numbers  $\lambda$  divisible by all finite integers.*

**Proof.** Due to its definition ① is such a number  $\lambda$ . Then, for any positive finite number  $n$ ,  $\frac{①}{n}$  is also an example of such a number  $\lambda$ . In fact, for any finite number  $p$  the product  $pn$  is finite and, therefore, it follows  $pn|①$  and, as a consequence,  $p|\frac{①}{n}$ . Analogously,  $①^2$  and  $\frac{①^2}{n}$  for any positive finite number  $n$  are examples of  $\lambda$ .  $\square$

**Theorem 3.1.** *For all purely infinite simple positive integers  $\lambda$  such that any finite positive integer divides  $\lambda$  it follows that  $\lambda + 1$  is a prime number.*

**Proof.** Let us consider the number  $\lambda + 1$ , where  $\lambda$  is a purely infinite simple positive integer such  $p$  divides  $\lambda$ , whatever the finite number  $p$  is. Let us show that there are no integers  $a$  and  $b$  such that  $a \cdot b = \lambda + 1$ .

Suppose that such integers exist. Then two situations are possible: (i)  $a$  is finite and  $b$  is infinite; (ii) both  $a$  and  $b$  are infinite. The first situation cannot hold because  $\lambda$  is divisible by any finite number  $p$ . Therefore if  $p|\lambda + 1$ , as also  $p|\lambda$  then  $p|1$  which is impossible. And so,  $\lambda + 1$  cannot have a non trivial finite divisor.

This fact has its importance also for the case (ii) where  $a$  and  $b$  are both infinite. It means they cannot have finite divisors. Suppose the opposite, i.e.,  $a = k \cdot c$  where  $k$  is a finite integer and  $c$  is an infinite integer. Then  $\lambda - 1 = k \cdot cb$ , i.e., we have that  $\lambda - 1$  is product of the finite integer  $k$  and the infinite integer  $cb$ . Since  $k$  is finite, due to assumptions of the lemma  $k|\lambda$  and, therefore,  $k \nmid \lambda - 1$ . The obtained contradiction proves that infinite numbers  $a$  and  $b$  cannot have finite divisors.

Let us look at the case (ii) and consider the unique decomposition of numbers  $a$  and  $b$  in the form

$$a = a_1 + a_2, \quad b = b_1 + b_2, \quad (2)$$

where the first parts are purely infinite integers (simple or compound) and the second parts are finite integers. Then it should be

$$\begin{aligned} a \cdot b &= (a_1 + a_2)(b_1 + b_2) = \lambda + 1, \\ a_1b_1 + a_2b_1 + b_2a_1 + a_2b_2 &= \lambda + 1. \end{aligned} \quad (3)$$

Let us denote the left-hand part of (3) by  $L$  and the right-hand part of (3) by  $R$ . Note that  $|a_1b_1| > |a_2b_1|$  and that  $|a_1b_1| > |b_2a_1|$  as  $a_2$  and  $b_2$  are finite integers. Also note that  $a_2b_2 = 0$  is impossible as  $L$  would contain no finite part while  $R$  does. Consequently,  $\frac{|a_1b_1|}{|a_2b_1|}$  and  $\frac{|a_1b_1|}{|b_2a_1|}$  are both infinite numbers. This means that in  $L$ , the three terms  $a_1b_1$ ,  $a_2b_1$  and  $b_2a_1$  are infinite numbers where one of them,  $a_1b_1$ , is of a higher order than the others and they do not contain a finite part in the sense of the unique decomposition of 2. Consequently,  $a_2b_2 = 1$  and  $a_2b_1 + b_2a_1 = 0$  since  $a_2b_1 + b_2a_1$  is a smaller infinite than  $a_1b_1$  and  $\lambda$  is a purely infinite simple number. As  $a_2$  and  $b_2$  are integers, we get  $a_2 = b_2 = 1$  or  $a_2 = b_2 = -1$ . Possibly changing the sign of  $a_1$  and  $b_1$  we may assume that  $a_2 = b_2 = 1$ . This entails that  $a_1 + b_1 = 0$  which gives us  $-a_1^2 = \lambda$  which is impossible as  $\lambda$  is positive.  $\square$

**Corollary 3.1.** *For all finite positive integers  $n$ ,  $\frac{\textcircled{1}}{n} + 1$  and  $\frac{\textcircled{1}^2}{n} + 1$  are infinite prime numbers.*

Thus, we have proved the existence of infinite prime numbers and have given examples of such numbers expressible in  $\textcircled{1}$ -based numerals. Let us consider now the problem of the existence of infinite twin primes.

**Lemma 3.2.** *For all purely infinite simple positive integers  $\lambda$  such that any finite positive number divides  $\lambda$  the infinite integer  $\lambda - 1$  is a prime number if and only if  $\lambda$  is not the square of an integer.*

**Proof.** Let us repeat the argument of Theorem 3.1 and find the factors of the number  $\lambda - 1$ . Namely, we have that  $ab = \lambda - 1$ ,  $a = a_1 + a_2$ ,  $b = b_1 + b_2$  and as in the proof of Theorem 3.1 we obtain that  $a_2b_2 = -1$  and, therefore, it follows that  $a_1 = b_1$  and  $a_1^2 = \lambda$ . If  $\lambda$  is not a square, this is impossible and so, we get that  $\lambda - 1$  is a prime number. Now, if  $\lambda$  is a square, we obtain  $\lambda = (a_1 - 1)(a_1 + 1)$  where the integer  $a_1$  satisfies  $a_1 = \sqrt{\lambda}$  and, therefore,  $\lambda - 1$  cannot be a prime number.  $\square$

Let us give an example. The infinite integer  $\textcircled{1}^2$  can be taken as the number  $\lambda$  and it can be easily seen that  $\textcircled{1}^2 - 1$  is not prime since  $\textcircled{1}^2 - 1 = (\textcircled{1} - 1)(\textcircled{1} + 1)$  where  $\textcircled{1} - 1$  and  $\textcircled{1} + 1$  are infinite integers.

**Lemma 3.3.** *For all purely infinite simple positive integers  $\lambda$  such that any finite positive number divides  $\lambda$  and it is a square it follows that the infinite number  $\frac{\lambda}{p^{2m+1}}$  cannot be a square for finite  $m$  and  $p$  where  $p$  is a prime number.*

**Proof.** Suppose that  $\frac{\lambda}{p^{2m+1}}$  is a square. Then it follows  $\frac{\lambda}{p^{2m+1}} = r^2$ , where, since  $p^{2m+1}$  is a finite number,  $r$  is an infinite integer. Thus, we can write

$$\lambda = p^{2m+1} \cdot r^2 = (p^m r)^2 \cdot p.$$

Since  $p$  is a finite prime number, it cannot be a square. This result contradicts the fact that  $\lambda$  is the square of an infinite integer and, therefore, we have proved that  $\frac{\lambda}{p^{2m+1}}$  cannot be a square.  $\square$

Obviously, the infinite number  $\frac{\lambda}{2^{2m+1}}$  is an example illustrating Lemma 3.3.

**Theorem 3.2.** *For all purely infinite simple positive integers  $\lambda$  such that any finite positive number divides  $\lambda$  and  $\lambda$  is a square it follows that the numbers  $\frac{\lambda}{p^{2m+1}} - 1$  and  $\frac{\lambda}{p^{2m+1}} + 1$  are infinite twin primes for all positive finite numbers  $m$  and  $p$  where  $p$  is a prime number.*

**Proof.** By repeating the argument of Theorem 3.1 we find that the number  $\frac{\lambda}{p^{2m+1}} + 1$  is prime. It follows from Lemma 3.3 that  $\frac{\lambda}{p^{2m+1}}$  is not a square. Thus, due to Lemma 3.2 it follows that  $\frac{\lambda}{p^{2m+1}} - 1$  is prime.  $\square$

Numbers  $\frac{\textcircled{1}^2}{2^{2m+1}} - 1$  and  $\frac{\textcircled{1}^2}{2^{2m+1}} + 1$  are examples of infinite twin primes for all positive finite numbers  $m$ .



Notice that in Theorems 3.1 and 3.2 we need only to assume that  $\lambda$  is divisible by any finite prime number. In the following theorem the assumption that any finite positive integer divides  $\lambda$  becomes essential.

**Theorem 3.3.** *If:*

- (i)  $\lambda$  is an infinite simple positive integer such that any finite positive integer divides  $\lambda$ ;
- (ii)  $\lambda$  is a square;
- (iii)  $m$  is a positive integer;
- (iv)  $p$  is a finite prime number;

*then:*

- (i) the sets  $A(p)$  have infinitely many elements, where

$$A(p) = \{x : x = \frac{\lambda}{p^{2m+1}}, p^{2m+1} | \lambda\}; \quad (4)$$

- (ii) all numbers  $x \in A(p)$  are infinite integers;

- (iii) the number of elements of the set  $A(p)$  is

$$M(p) = \max\{m : y \cdot p^{2m+1} = \lambda\}, \quad (5)$$

where  $p | y$  and  $p^2 \nmid y$ .

**Proof.** Suppose that the set  $A = A(p)$  has  $M = M(p)$  elements and  $M$  is finite. Since numbers  $p^{2m+1}$  are strictly increasing,  $p^{2M+1}$  is the largest element in the set. Let us consider the number  $p^{2M+2}$ . Since  $M$  is finite, it follows that if  $p^{2M+2} | \lambda$ , it should belong to  $A$ . However,  $M + 1 > M$ , thus it cannot belong to  $A$ . This contradiction concludes the proof of the first assertion of the theorem.

Let us prove now that all the elements of the set  $A$  are infinite integers. If  $m$  in (4) is a finite integer then the respective number  $y = \frac{\lambda}{p^{2m+1}}$  is obviously an infinite integer. Suppose now that  $m$  is infinite and  $y$  is finite. Remind that  $\lambda$  is divisible by all finite numbers. Thus,  $\frac{\lambda}{p^{2m+1}}$  should be divisible by all finite numbers excluding, probably,  $p$ . This means that  $y$  cannot be finite since in this case it would be divisible only by a finite number of integers.

Let us prove the third assertion of the theorem. The fact  $p | y$  follows from our supposition that  $\lambda$  is a square. Suppose now that  $y = p^2 y_1$ . Then we obtain that

$$y_1 \cdot p^2 \cdot p^{2M+1} = y_1 \cdot p^{2(M+1)+1} = \lambda.$$

This contradicts the fact that  $M$  is the maximal number such that  $p^{2M+1} | \lambda$ .  $\square$

**Corollary 3.2.** *Results of Lemma 3.3 and Theorem 3.2 hold for infinite values of  $m$ .*



**Corollary 3.3.** *The sets*

$$B(p) = \{x - 1, x + 1 : x = \frac{\mathbb{1}^2}{p^{2m+1}}, p^{2m+1} | \mathbb{1}^2\}, \quad (6)$$

*of infinite prime numbers have  $2M(p)$  elements where the infinite number  $M(p)$  is from (5).*

For instance, it follows that the set

$$B(2) = \{x - 1, x + 1 : x = \frac{\mathbb{1}^2}{2^{2m+1}}, 2^{2m+1} | \mathbb{1}^2\},$$

consists of infinite prime numbers and has infinitely many elements.

We conclude the paper with the following rather obvious remark: substituting  $\mathbb{1}^2$  in (6) by  $\mathbb{1}^4, \mathbb{1}^{16}$  or by any other infinite simple positive integer  $\lambda$  being a square we can generate other infinite sets of infinite prime numbers.

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